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Game Theory

Game theory is useful for making decisions in cases where two or more decision-makers have conflicting interests. Our lecture deals with situations where there are only two decision-makers (or players). How to select the optimal strategy for each player without knowledge of the competitor's strategy is the basic problem of playing a game? The theory of games is based on the minimax principle which implies that each competitor will act to minimize has a maximum loss (or maximize his minimum gain (maxmin principle)) or achieve the best of the worst.

نظرية الالعاب مفيدة لاتخاذ القرارات في الحالات التي يكون فيها اثنان أو أكثر من صناع القرار لديهم مصالح متضاربة. تتناول محاضرتنا المواقف التي لا يوجد فيها سوى اثنين من صناع القرار (أو اللاعبين). كيف تختار الإستراتيجية المثلى لكل لاعب بدون معرفة إستراتيجية المنافس هي المشكلة الأساسية للعب اللعبة؟ تعتمد نظرية الألعاب على مبدأ minimax الذي يعني أن كل منافس سيعمل على تقليل الخسارة القصوى (أو زيادة الحد الأدنى من مكاسبه maxmin) أو تحقيق أفضل الأسوأ

Some important definitions:

1- **Games:** A game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person gets some benefit or suffers a loss.

تُعرَّف اللعبة بأنها نشاط بين شخصين أو أكثر تتضمن أنشطة كل شخص وفقًا لمجموعة من القواعد، وفي نهاية المطاف يحصل كل شخص على بعض الفوائد أو يعاني من خسارة

2- **Play:** The set of rules defines the game; going through the set of rules once by participants defines a play.

تمر مجموعة القوانين التي تعرف اللعبة من خلال مجموعة من القواعد مرة واحدة من قبل المشاركين

3- **Game of strategy:** If in a game, activities are determined by skill, it is said to be a game of strategy; if they are determined by chance, it is a **Game of Chance.**

اذا كانت الأنشطة في لعبة ما تحددها المهارة ، فيُقال إنها لعبة استراتيجية ؛ إذا تم تحديدهم بالصدفة ، فهي لعبة فرصة.

4- **Two-person game:** Suppose there is a number (n) of players such that $n \geq 2$. In the case of $n = 2$, it is called a Two-person game.

5- **Strategy:** A strategy for a given player is a set of rules that specifies which of the available courses of action he should make at each play.

الإستراتيجية الخاصة بلاعب معين هي مجموعة من القواعد التي تحدد مسارات العمل المتاحة التي يجب أن يقوم بها في كل لعبة

There are two types of strategy

i- Pure strategy: A pure strategy is a predetermined plan that describes for a player the sequence of moves and countermoves حركات مضادة he will make during a complete game. Palyer A(B) knows player B's(A's) set, but he does not know for sure which element of the set B(A) has picked at the commencement بداية of a given play of the game. The objective is find best strategy for a given player. If players know exactly what the other player is going to do, a deterministic situation حالة محددة is obtained and the objective is to maximize the gain.

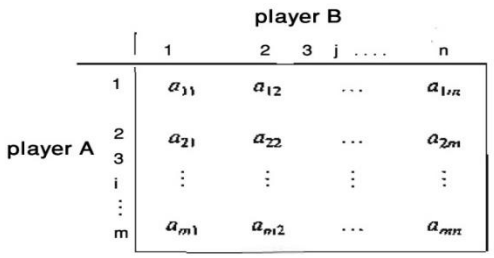
ii- Mixed strategy: If a player is guessing يخزر اللاعب as to which activity is to be selected by the other player, a probabilistic situation حالة محتملة is obtained نحصل على وضع محتمل and the objective function is to maximize the expected gain.

6- **Two people Zero-Sum Games (Rectangular Games):** A game with only two players (say A and B). If the losses of one player are equivalent to the gains of the other, so that the sum of their net gains is zero. The resulting gain is represented by pay-off-matrix in rectangular form.

Example1: Assume we are playing ROCK-SCISSORS-PAPER for one dollar. Then the payoff matrix is

	Rock	Scissors	Paper
Rock	0	1	−1
Scissors	−1	0	1
Paper	1	−1	0

7- **Pay-off-matrix (gain matrix):** it is a table that shows how payment should be made at the end of a play. The representation indicates that if A uses strategy i and B uses strategy j, the payoff to A is a_{ij}, which means that the payoff to B is $-a_{ij}$.



8- **Saddle point:** It is a position of such an element in the pay-off matrix, which is the **minimum in its row and maximum in its column**. The payoff at the saddle point is called the **value of the game**.

Example2: (Chicken الجبان). Consider the following two-player game: Two cars face each other and begin driving (quickly) toward each other. The player who swerves ينحرف first loses 1 point, the other player wins 1 point. If both players

swerve, then each receives 0 points. If neither player swerves, a very bad crash occurs and both players lose 10 points. Assuming that the strategies for Player 1 are in the rows, while the strategies for Player 2 are in the columns, then the two matrices for the players are:



	Swerve	Don't Swerve
Swerve	0	-1
Don't Swerve	1	-10

	Swerve	Don't Swerve
Swerve	0	1
Don't Swerve	-1	-10

9- Rule of Dominance:

a- If all elements of a row, say ith, are less than or equal to the corresponding elements of any other row, say jth row, then ith row is dominated by jth row.

b- If all elements of a column, say ith, are greater than or equal to the corresponding elements of any other column, say jth, then ith column is dominated by jth column.

c- A pure strategy may be dominated if it is inferior أدنى to an average of two or more other pure strategies.

Pure Strategies: Games with Saddle Point (Stable games)

Player A whose strategies are repressed by the rows. Player B whose strategies are repressed by columns. They do it as

1- Player A chooses minimum value in each row (least gain or payoff to player A), then he chooses maximin strategy to maximize his minimum gains.

2- Player B wants to minimize his losses, so he chooses maximum loss value in each column, and then he chooses a minimax strategy to minimize his maximum losses.

If the **maximin value equals the minimax value**, then the game is said to have a **saddle point** and the corresponding strategies are called optimal strategies.

Example3: Solve the game whose payoff matrix is given by.

		B		
		I	II	III
A	I	-2	15	-2
	II	-5	-6	-4
	III	-5	20	-8

Solution:

A \ B	I	II	III	
I	-2	15	-2	-2
II	-5	-6	-4	-6
III	-5	20	-8	-8
	-2	20	-2	

1- The best strategy for A is I, min {I, II, III} ={-2, -6, -8}, max {-2, -6, -8} = -2
2- The best strategy for B is either I or II, max {I, II, III}={-2, 20, -2}, min{-2, 20, -2} = -2
3- The optimal value of the game is -2 for A and -2 for B.

$$Maxmin = -2 = minmax$$

Then G^* = -2 which is a saddle point.

Example4: Solve the following game whose payoff matrix is given by.

A \ B	I	II	III
I	6	8	6
II	4	12	2

Solution:
1- The best strategy for A is …6…….
2- The best strategy for B is …6……
3- The value of the game is ……6….. for A and ……6… for B.

$$Maxmin = 6 = minmax$$

G^*= 6 saddle point

H.W: Solve the following game whose payoff matrix is given by.

		B		
		I	II	III
A	I	-3	-2	6
	II	2	0	2
	III	5	-2	-4

Mixed Strategies

Whenever a game does not produce a saddle point, game theory advises each player to assign a probability distribution over their set of strategies. A mixed strategy is a selection among pure strategies with some fixed probabilities (proportions). A pure strategy is a special case of a mixed strategy.

عندما لا تنتج اللعبة نقطة سرج، تنصح نظرية اللعبة كل لاعب بتعيين توزيع احتمالي على مجموعة الاستراتيجيات الخاصة بهم. الإستراتيجية المختلطة هي الاختيار بين الاستراتيجيات البحتة مع بعض الاحتمالات الثابتة (النسب). الاستراتيجية الخالصة هي حالة خاصة للاستراتيجية المختلطة.

Define

x_i = probability that player 1 will use strategy i (i = 1, 2, . . . , m),

y_j = probability that player 2 will use strategy j (j = 1, 2, . . . , n),

If $x_1 \geq 0, x_2 \geq 0, and\ x_1 + x_2 = 1$, (x_1, x_2) is a mixed strategy. For example, the mixed strategy $(\frac{1}{2}, \frac{1}{2})$. Similarly $y_1 \geq 0, y_2 \geq 0, and\ y_1 + y_2 = 1$, (y_1, y_2) is a mixed strategy.

Any mixed strategies $(x_1, x_2, \ldots, x_m)$ for the row player is a pure strategy if any of the strategy x_i equals 1. Similarly, any mixed strategies $(y_1, y_2, \ldots, y_n)$ for column player is a pure strategy if any of the strategy y_i equal 1, where m and n are the respective numbers of available strategies.

Thus, player 1 would specify his plan for playing the game by assigning values to $x_1, x_2, \ldots, x_m$, because these values are probabilities. Similarly, the plan for player 2 would be described by the values he assigns to his decision variables $y_1, y_2, \ldots, y_n$.

Mixed Strategy: Games without saddle point (unstable games)

Pure strategies are available as optimal strategies only for that game theory that has a saddle point. For games that do not have a saddle point can be solved by applying the concept of mixed strategies. A mixed strategy game can be solved by:

تتوفر الاستراتيجيات البحتة كاستراتيجيات مثالية فقط لنظرية اللعبة التي لها نقطة سرج. بالنسبة للألعاب التي لا تحتوي على نقطة سرج يمكن حلها من خلال تطبيق مفهوم الاستراتيجيات المختلطة. يمكن حل لعبة الإستراتيجية المختلطة من خلال:

(i) Algebraic method.
(ii) Graphical method.
(iii) Linear programming method.

By Using Algebraic Method

Let player A selects strategy A_1 with a probability of x_1 and therefore selects strategy A_2 with a probability of x_2.

Let player B selects strategy B_1 with a probability of y_1 and therefore selects strategy B_2 with a probability of y_2.

Theorem: For any 2×2 two people zero-game without any saddle point having pay-off matrix for player A is

		B	
		$B_1(y_1)$	$B_2(y_2)$
A	$A_1(x_1)$	a_{11}	a_{12}
	$A_2(x_2)$	a_{21}	a_{22}

The optimal strategies (x_1^*, x_2^*) and (y_1^*, y_2^*) are given by $\frac{x_1}{x_2} = \frac{a_{22}-a_{21}}{a_{11}-a_{12}}$ and $\frac{y_1}{y_2} = \frac{a_{22}-a_{12}}{a_{11}-a_{21}}$ and the optimal value to the game is given by $G^* = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$.

Proof: We get from the proof that from using $x_1 + x_2 = 1$, we get

$$\frac{x_1}{x_2} + \frac{x_2}{x_2} = \frac{1}{x_2}$$

$$\frac{a_{22} - a_{21}}{a_{11} - a_{12}} + 1 = \frac{1}{x_2}$$

$$\frac{(a_{22} - a_{21}) + (a_{11} - a_{12})}{(a_{11} - a_{12})} = \frac{1}{x_2}$$

$$x_2 = \frac{(a_{11} - a_{12})}{(a_{22} - a_{21}) + (a_{11} - a_{12})}$$

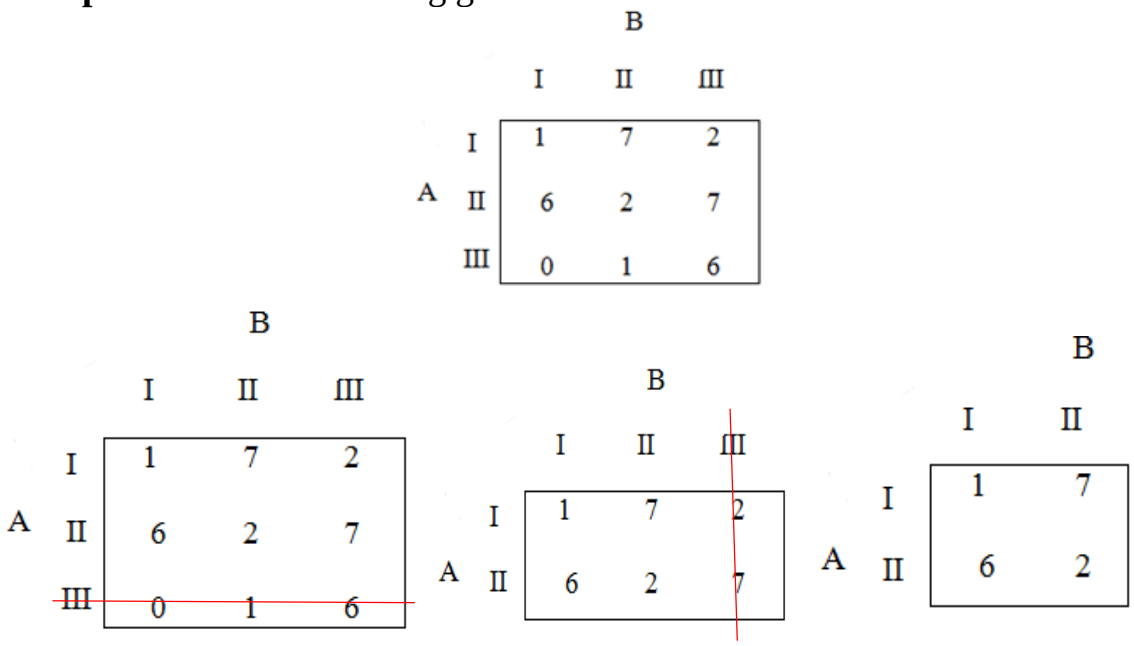
In the same way we get the rest

$$x_1^* = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}, \quad x_2^* = \frac{a_{11}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

And by using $y_1 + y_2 = 1$ we get

$$y_1^* = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}, \quad y_2^* = \frac{a_{11}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

Example: Solve the following game.



Solution:

Step1: Row3 is less than row2, so we delete row3
Step2: Column III is dominated by column I, so we delete III
We try to find saddle point if it is not exist then solve it by algebraic method.
Then

$$x_1^* = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-6}{(1+2)-(7+6)} = \frac{-4}{-10} = \frac{2}{5},$$
$$x_2^* = \frac{a_{11}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{1-7}{(1+2)-(7+6)} = \frac{-6}{-10} = \frac{3}{5}$$

And by using $y_1 + y_2 = 1$ we get

$$y_1^* = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-7}{(1+2)-(7+6)} = \frac{-5}{-10} = \frac{1}{2}$$
$$y_2^* = \frac{a_{11}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{1-6}{(1+2)-(7+6)} = \frac{-5}{-10} = \frac{1}{2}$$
$$G^* = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{1\times2-7\times6}{(1+2)-(7+6)} = 4$$

So we obtain that

A chooses optimal strategies $(x_1^*, x_2^*, x_3^*) = (\frac{2}{5}, \frac{3}{5}, 0)$, **B** chooses optimal strategies $(y_1^*, y_2^*, y_3^*) = (\frac{1}{2}, \frac{1}{2}, 0)$ and the optimal value of the game is 4.

$$Maxmin = 2 \neq minmax = 6$$
$$Maxmin = 2 \leq 4 \leq minmax = 6$$

Example: Solve the game whose pay off matrix is given by

Player A \ Player B	B_1	B_2
A_1	2	-1
A_2	-1	1

Player A \ Player B	B_1	B_2	Row Minimum
A_1	2	[-1]	[- 1]
A_2	-1	(1)	-1
Column Maximum	2	(1)	

This game has no saddle point.

$$Maxmin = -1 \neq minmax = 1$$

Solution using algebraic method

Here $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -1$, $a_{22} = 1$

$$x_1^* = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{1+1}{(2+1)-(-1-1)} = \frac{2}{3+2} = \frac{2}{5},$$

$$x_2^* = 1 - x_1^* = 1 - \frac{2}{5} = \frac{3}{5}$$

And by using $y_1 + y_2 = 1$ we get

$$y_1^* = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{1+1}{(2+1)-(-1-1)} = \frac{2}{3+2} = \frac{2}{5}$$

$$y_2^* = 1 - y_1^* = 1 - \frac{2}{5} = \frac{3}{5}$$

$$G^* = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2\times1-(-1\times-1)}{(2+1)-(-1-1)} = \frac{2-1}{3+2} = \frac{1}{5}$$

A chooses optimal strategies $(x_1^*, x_2^*) = (\frac{2}{5}, \frac{3}{5})$, **B** chooses optimal strategies $(y_1^*, y_2^*) = (\frac{2}{5}, \frac{3}{5})$ and the optimal value of the game is 1/5.

$$Maxmin = -1 \leq 1/5 \leq minmax = 1$$

H.W2: Solve the following game whose payoff matrix is given by the matrix down Using Algebraic Method.

		B		
		I	II	III
A	I	-3	-2	6
	II	2	0	2
	III	5	-2	-4

H.W3: Solve the game whose pay off matrix is given by

		B		
		I	II	III
A	I	- 5	10	20
	II	5	-10	-10
	III	5	-20	-20

References:

1- Wayne L. Winston, Operations Research Applications and Algorithms, 1991.
2- B. N. Mishra, Optimization Linear Programming, Ane's Student Edition, 2012.
https://youtu.be/o75FL98unhY
https://youtu.be/lxz7Wq2cvEY

2. Dominance rule to reduce the size of the payoff matrix
Using dominance property

		Player B		
		B_1	B_2	B_3
Player A	A_1	10	5	-2
	A_2	13	12	15
	A_3	16	14	10

2. Dominance rule to reduce the size of the payoff matrix
Using dominance property

		Player B			
		B_1	B_2	B_3	B_4
Player A	A_1	10	5	-2	4
	A_2	13	12	15	11
	A_3	16	14	10	10
	A_4	17	13	15	11

Find Solution of game theory problem using saddle point

Player A\Player B	B1	B2	B3	B4
A1	20	15	12	35
A2	25	14	8	10
A3	40	2	10	5
A4	-5	4	11	0

Find Solution of game theory problem using algebraic method

Player A\Player B	B1	B2
A1	1	7
A2	6	2